

ON THE AUTOMORPHISM GROUPS OF MODELS IN \mathbb{C}^2

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ABSTRACT. In this note, we consider models in \mathbb{C}^2 . The purpose of this note is twofold. We first show a characterization of models in \mathbb{C}^2 by their noncompact automorphism groups. Then we give an explicit description for automorphism groups of models in \mathbb{C}^2 .

1. INTRODUCTION

For a domain Ω in the complex Euclidean space \mathbb{C}^n , the set of biholomorphic self-maps forms a group under the binary operation of composition of mappings, which is called *automorphism group* ($\text{Aut}(\Omega)$). The topology on $\text{Aut}(\Omega)$ is that of uniform convergence on compact sets (i.e., the compact-open topology).

A boundary point $p \in \partial\Omega$ is called a *boundary orbit accumulation point* if there exist a sequence $\{f_j\} \subset \text{Aut}(\Omega)$ and a point $q \in \Omega$ such that $f_j(q) \rightarrow p$ as $j \rightarrow \infty$. The classification of domains with noncompact automorphism groups is pertinent to the study of the geometry of the boundary at an orbit accumulation point.

In this note, we consider a model

$$M_H = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + H(z_1) < 0\},$$

where H is a homogeneous subharmonic polynomial of degree $2m$ ($m \geq 1$) which contains no harmonic terms. It is a well-known result of F. Berteloot [7] that if $\Omega \subset \mathbb{C}^2$ is pseudoconvex, of D'Angelo finite type near a boundary orbit accumulation point, then Ω is biholomorphically equivalent to a model M_H . For the case Ω is strongly pseudoconvex, this result was proved by B. Wong [28] and J. P. Rosay [29]; indeed, the model is biholomorphically equivalent to the unit ball. These results motivate the following several concepts.

A domain $\Omega \subset \mathbb{C}^2$ is said to satisfy *Condition (M)* at $p \in \partial\Omega$ if there exist neighborhoods U and V of p and $(0, 0)$, respectively; a biholomorphism Φ from $U \cap \Omega$ onto $V \cap M_H$, which extends homeomorphically to $U \cap \partial\Omega$ such that $\Phi(p) = (0, 0)$. In this circumstance, we say that a sequence $\{\eta_n\} \subset U \cap \Omega$ *converges tangentially to order s ($s > 0$)* to p if $\text{dist}(\Phi(\eta_n), \partial M_H) \approx |\Phi(\eta_n)_1|^s$, where $\text{dist}(z, \partial M_H)$ is the Euclidean distance from z to ∂M_H and $\Phi(\eta_n)_1$ is the first coordinate of $\Phi(\eta_n)$. Here and in what follows, \lesssim and \gtrsim denote inequality up to a positive constant. Moreover, we will use \approx for the combination of \lesssim and \gtrsim .

We first prove the following theorem.

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Theorem 1. *Let Ω be a domain in \mathbb{C}^2 and let $p \in \partial\Omega$. Suppose that Ω satisfies Condition (M) at p and there exist a sequence $\{f_n\} \subset \text{Aut}(\Omega)$ and $q \in \Omega$ such that $\{f_n(q)\}$ converges tangentially to order $\leq 2m$ ($= \deg(H)$) to p . Then Ω is biholomorphically equivalent to the model M_H .*

For a domain Ω in \mathbb{C}^n , the automorphism group is not easy to describe explicitly; besides, it is unknown in most cases. For instance, the automorphism groups of various domains are given in [10, 15, 19, 20, 22, 25, 26]. Recently, explicit forms of automorphism groups of certain domains have been obtained in [1, 8, 9].

The second part of this note is to describe automorphism groups of models in \mathbb{C}^2 . If a model is symmetric, i.e. $H(z_1) = |z_1|^{2m}$, then it is biholomorphically equivalent to the Thullen domain $E_{1,m} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2|^2 + |z_1|^{2m} < 1\}$; the $\text{Aut}(E_{1,m})$ is exactly the set of all biholomorphisms

$$(z_1, z_2) \mapsto \left(e^{i\theta_1} \frac{z_1 - a}{1 - \bar{a}z_1}, e^{i\theta_2} \frac{(1 - |a|^2)^{1/2m}}{(1 - \bar{a}z_1)^{1/m}} z_2 \right)$$

for some $a \in \mathbb{C}$ with $|a| < 1$ and $\theta_1, \theta_2 \in \mathbb{R}$ (cf. [15, Example 9, p.20]). Let us denote by $\Omega_m = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + (\text{Re } z_1)^{2m} < 0\}$. All the other models, which are not biholomorphically equivalent to $E_{1,m}$ or Ω_m , will be treated together, as the generic case. Let us denote $T_t^1, T_t^2, R_\theta, S_\lambda$ by the following automorphisms:

$$\begin{aligned} T_t^1 : (z_1, z_2) &\mapsto (z_1 + it, z_2); \\ T_t^2 : (z_1, z_2) &\mapsto (z_1, z_2 + it); \\ R_\theta : (z_1, z_2) &\mapsto (e^{i\theta} z_1, z_2); \\ S_\lambda : (z_1, z_2) &\mapsto (\lambda z_1, \lambda^{2m} z_2), \end{aligned}$$

where $t \in \mathbb{R}$, $\lambda > 0$, and $\exp(i\theta)$ is an L -root of unity (see Section 4).

With these notations, we obtain the following our second main result.

Theorem 2. *If $m \geq 2$, then*

(i) *$\text{Aut}(\Omega_m)$ is generated by*

$$\{T_t^1, T_t^2, R_\pi, S_\lambda \mid t \in \mathbb{R}, \lambda > 0\};$$

(ii) *For any generic model M_H , $\text{Aut}(M_H)$ is generated by*

$$\{T_t^2, R_\theta, S_\lambda \mid t \in \mathbb{R}, \lambda > 0, \text{ and } \exp(i\theta) \text{ is an } L\text{-root of unity}\}.$$

Let $S(\Omega)$ denote the set of all boundary accumulation points for $\text{Aut}(\Omega)$. Then it follows from Theorem 2 that

- (i) $S(E_{1,m}) = \{(e^{i\theta}, 0) \in \mathbb{C}^2 : \theta \in [0, 2\pi)\}$;
- (ii) $S(\Omega_m) = \{(it, is) \in \mathbb{C}^2 : t, s \in \mathbb{R}\} \cup \{\infty\}$;
- (iii) $S(M_H) = \{(it, 0) \in \mathbb{C}^2 : t \in \mathbb{R}\} \cup \{\infty\}$ for any generic model M_H .

We remark that, for any model M_H in \mathbb{C}^2 , $S(M_H)$ is a smooth submanifold of ∂M_H . Moreover, the D'Angelo type is constant and maximal along $S(M_H)$. In addition, the behaviour of orbits in any model $M_H \subset \mathbb{C}^2$ is well-known. For instance, if there exist a point $q \in M_H$ and a sequence $\{f_n\} \subset \text{Aut}(M_H)$ such that $\{f_n(q)\}$ converges to some boundary accumulation point $p \in S(M_H) \setminus \{\infty\}$, then it must converge tangentially to order $\leq \deg(H)$ to p . In the past twenty years, much attention has been given to the behaviour of orbits near an orbit accumulation point. We refer the reader to the articles [16, 18, 17], and references therein for the development of related subjects.

A typical consequence of Theorem 2 and the Berteloot's result [7] is as follows.

Corollary 1. *Let Ω be a domain in \mathbb{C}^2 . Suppose that there exist a point $q \in \Omega$ and a sequence $\{f_j\} \subset \text{Aut}(\Omega)$ such that $\{f_j(q)\}$ converges to $p_\infty \in \partial\Omega$. Assume that the boundary of Ω is smooth, pseudoconvex, and of D'Angelo finite type near p_∞ ($\tau(\partial\Omega, p_\infty) = 2m$). Then exactly one of the following alternatives holds:*

(i) *If $\dim \text{Aut}(\Omega) = 2$ then*

$$\Omega \simeq M_H = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + H(z_1) < 0\},$$

where M_H is a generic model in \mathbb{C}^2 and $\deg(H) = 2m$.

(ii) *If $\dim \text{Aut}(\Omega) = 3$ then*

$$\Omega \simeq \Omega_m = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + (\text{Re } z_1)^{2m} < 0\}.$$

(iii) *If $\dim \text{Aut}(\Omega) = 4$ then*

$$\Omega \simeq E_{1,m} = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + |z_1|^{2m} < 0\} \simeq \{(z_1, z_2) \in \mathbb{C}^2 : |z_2|^2 + |z_1|^{2m} < 1\}.$$

(iv) *If $\dim \text{Aut}(\Omega) = 8$ then*

$$\Omega \simeq \mathbb{B}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}.$$

The dimensions 0, 1, 5, 6, 7 cannot occur with Ω as above.

For the case that $\partial\Omega$ is real analytic and of D'Angelo finite type near a boundary orbit accumulation point (without the hypothesis of pseudoconvexity), a similar result as the above corollary was obtained in [27] by using a different method. In addition, it was shown in [3] that a smoothly bounded Ω in \mathbb{C}^2 with real analytic boundary and with noncompact automorphism group, must be biholomorphically equivalent to $E_{1,m}$.

This paper is organized as follows. In Section 2, we review some basic notions needed later. In Section 3, we prove Theorem 1. Finally, the proof of Theorem 2 is given in Section 4

2. DEFINITIONS AND RESULTS

First of all, we recall the following definitions.

Definition 1 (see [11]). Let $\Omega \subset \mathbb{C}^n$ be a domain with \mathcal{C}^∞ -smooth boundary and $p \in \partial\Omega$. Then the D'Angelo type $\tau(\partial\Omega, p)$ of $\partial\Omega$ at p is defined as

$$\tau(\partial\Omega, p) := \sup_{\gamma} \frac{\nu(\rho \circ \gamma)}{\nu(\gamma)},$$

where ρ is a defining function of Ω near p , the supremum is taken over all germs of nonconstant holomorphic curves $\gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, p)$. We say that p is a point of *finite type* if $\tau(\partial\Omega, p) < \infty$ and of *infinite type* if otherwise.

Definition 2. Let X, Y be complex spaces and $\mathcal{F} \subset \text{Hol}(X, Y)$.

- (i) A sequence $\{f_j\} \subset \mathcal{F}$ is *compactly divergent* if for every compact set $K \subset X$ and for every compact set $L \subset Y$ there is a number $j_0 = j_0(K, L)$ such that $f_j(K) \cap L = \emptyset$ for all $j \geq j_0$.
- (ii) The family \mathcal{F} is said to be *not compactly divergent* if \mathcal{F} contains no compactly divergent subsequences.

Definition 3. A complex space X is called *taut* if for any family $\mathcal{F} \subset \text{Hol}(\Delta, X)$, there exists a subsequence $\{f_j\} \subset \mathcal{F}$ which is either convergent or compactly divergent, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

We recall the concept of Carathéodory kernel convergence of domains which is relevant to the discussion of scaling methods (see [13]). Note that the local Hausdorff convergence can replace the normal convergence in case the domains in consideration are convex.

Definition 4 (Carathéodory Kernel Convergence). Let $\{\Omega_\nu\}$ be a sequence of domains in \mathbb{C}^n such that $p \in \bigcap_{\nu=1}^{\infty} \Omega_\nu$. If p is an interior point of $\bigcap_{\nu=1}^{\infty} \Omega_\nu$, the Carathéodory kernel $\hat{\Omega}$ at p of the sequence $\{\Omega_\nu\}$, is defined to be the largest domain containing p having the property that each compact subset of $\hat{\Omega}$ lies in all but a finite number of the domains Ω_ν . If p is not an interior point of $\bigcap_{\nu=1}^{\infty} \Omega_\nu$, the Carathéodory kernel $\hat{\Omega}$ is $\{p\}$. The sequence $\{\Omega_\nu\}$ is said to converge to its kernel at p if every subsequence of $\{\Omega_\nu\}$ has the same kernel at p .

We shall say that a sequence $\{\Omega_\nu\}$ of domains in \mathbb{C}^n converges normally to $\hat{\Omega}$ (denoted by $\lim \Omega_\nu = \hat{\Omega}$) if there exists a point $p \in \bigcap_{\nu=1}^{\infty} \Omega_\nu$ such that $\{\Omega_\nu\}$ converges to its Carathéodory kernel $\hat{\Omega}$ at p .

Now we recall several results which will be used later on. The following proposition is a generalization of the theorem of Greene-Krantz [14] (cf. [23]).

Proposition 1. Let $\{A_j\}_{j=1}^{\infty}$ and $\{\Omega_j\}_{j=1}^{\infty}$ be sequences of domains in a complex manifold M with $\lim A_j = A_0$ and $\lim \Omega_j = \Omega_0$ for some (uniquely determined) domains A_0, Ω_0 in M . Suppose that $\{f_j : A_j \rightarrow \Omega_j\}$ is a sequence of biholomorphic maps. Suppose also that the sequence $\{f_j : A_j \rightarrow M\}$ converges uniformly on compact subsets of A_0 to a holomorphic map $F : A_0 \rightarrow M$ and the sequence $\{g_j := f_j^{-1} : \Omega_j \rightarrow M\}$ converges uniformly on compact subsets of Ω_0 to a holomorphic map $G : \Omega_0 \rightarrow M$. Then one of the following two assertions holds.

- (i) The sequence $\{f_j\}$ is compactly divergent, i.e., for each compact set $K \subset \Omega_0$ and each compact set $L \subset \Omega_0$, there exists an integer j_0 such that $f_j(K) \cap L = \emptyset$ for $j \geq j_0$, or
- (ii) There exists a subsequence $\{f_{j_k}\} \subset \{f_j\}$ such that the sequence $\{f_{j_k}\}$ converges uniformly on compact subsets of A_0 to a biholomorphic map $F : A_0 \rightarrow \Omega_0$.

In closing this section we recall the following lemma (see [7]).

Lemma 1 (F. Berteloot). Let σ_∞ be a subharmonic function of class \mathcal{C}^2 on \mathbb{C} such that $\sigma_\infty(0) = 0$ and $\int_{\mathbb{C}} \bar{\partial} \partial \sigma_\infty = +\infty$. Let $\{\sigma_k\}$ be a sequence of subharmonic functions on \mathbb{C} which converges uniformly on compact subsets of \mathbb{C} to σ_∞ . Let Ω be any domain in a complex manifold of dimension m ($m \geq 1$) and let z_0 be a fixed point in Ω . Denote by M_k the domain in \mathbb{C}^n defined by

$$M_k = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + \sigma_k(z_1) < 0\}.$$

Then any sequence $h_k \in \text{Hol}(\Omega, M_k)$ such that $\{h_k(z_0), k \geq 1\} \in M_\infty$ admits a subsequence which converges uniformly on compact subsets of Ω to an element of $\text{Hol}(\Omega, M_\infty)$.

3. ASYMPTOTIC BEHAVIOUR OF ORBITS IN A MODEL IN \mathbb{C}^2

Let P be a subharmonic polynomial. Let us denote by M_P the model given by

$$M_P = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) := \operatorname{Re} z_2 + P(z_1) < 0\}.$$

Let Ω be a domain in \mathbb{C}^2 . Suppose that $\partial\Omega$ is pseudoconvex, finite type, and smooth of class \mathcal{C}^∞ near a boundary point $p \in \partial\Omega$. In [7], F. Berteloot proved that if p is a boundary orbit accumulation point for $\operatorname{Aut}(\Omega)$, then Ω is biholomorphically equivalent to a model M_H , where H is a homogeneous subharmonic polynomial of degree $2m$ which contains no harmonic terms with $\|H\| = 1$. Here and in what follows, denote by $\|P\|$ the maximum of absolute values of the coefficients of a polynomial P . Let us denote by \mathcal{P}_{2m} the space of real valued polynomials on \mathbb{C} with degree less than or equals to $2m$ and which do not contain any harmonic term and by

$$\mathcal{H}_{2m} = \{H \in \mathcal{P}_{2m} \text{ such that } \deg(H) = 2m \text{ and } H \text{ is homogeneous and subharmonic}\}.$$

From now on, let $H \in \mathcal{H}_{2m}$ be as in Theorem 1. Taking the risk of confusion we employ the notation

$$H_j := \frac{\partial^j H}{\partial z_1^j}; \quad H_{j,\bar{q}} := \frac{\partial^{j+q} H}{\partial z_1^j \partial \bar{z}_1^q}$$

throughout the paper for all $j, q \in \mathbb{N}^*$.

For each $a = (a_1, a_2) \in \mathbb{C}^2$, let us define

$$H_a(w_1) = \frac{1}{\epsilon(a)} \sum_{j,q>0} \frac{H_{j,\bar{q}}(a_1)}{(j+q)!} \tau(a)^{j+q} w_1^j \bar{w}_1^q,$$

where $\epsilon(a) = |\operatorname{Re} a_2 + H(a_1)|$ and $\tau(a)$ is chosen so that $\|H_a\| = 1$. We note that $\sqrt{\epsilon(a)} \lesssim \tau(a) \lesssim \epsilon(a)^{1/(2m)}$. Denote by ϕ_a the holomorphic map

$$\begin{aligned} \phi_a : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ z &\mapsto w = \phi_a(z), \end{aligned}$$

given by

$$\begin{cases} w_2 = \frac{1}{\epsilon(a)} \left[z_2 - a_2 - \epsilon(a) + 2 \sum_{j=1}^{2m} \frac{H_j(a_1)}{j!} (z_1 - a_1)^j \right] \\ w_1 = \frac{1}{\tau(a)} [z_1 - a_1]. \end{cases}$$

It is easy to check that ϕ_a maps biholomorphically M_H onto M_{H_a} and $\phi_a(a) = (0, -1)$.

Now let us consider a domain Ω in \mathbb{C}^2 satisfying Condition (M) at a boundary point $p \in \partial\Omega$. With no loss of generality, we can assume $p = (0, 0)$ and

$$\Omega \cap U = \{(z_1, z_2) \in U : \rho(z_1, z_2) = \operatorname{Re} z_2 + H(z_1) < 0\}.$$

Assume that there exist a sequence $\{f_n\} \subset \operatorname{Aut}(\Omega)$ and a point $q \in M_H$ such that $\eta_n := f_n(q) \rightarrow (0, 0)$ as $n \rightarrow \infty$.

Remark 1. By Proposition 2.1 in [7], Ω is taut and after taking a subsequence we may assume that for each compact subset $K \subset D$ there exists a positive integer n_0 such that $f_n(K) \subset \Omega \cap U$ for every $n \geq n_0$.

Since $\|H_{\eta_n}\| = 1$, passing to a subsequence if necessary, we can assume that $\lim H_{\eta_n} = H_\infty$, where $H_\infty \in \mathcal{P}_{2m}$ and $\|H_\infty\| = 1$.

Proposition 2. Ω is biholomorphically equivalent to M_{H_∞} .

Proof. Let $\psi_n := \phi_{\eta_n} \circ f_n$ for each $n \in \mathbb{N}^*$ and consider the following sequence of biholomorphisms

$$\begin{aligned} \psi_n : f_n^{-1}(\Omega \cap U) &\rightarrow M_{H_{\eta_n}} \\ q &\mapsto (0, -1). \end{aligned}$$

By Lemma 1 and by Remark 1, after taking a subsequence we may assume that $\{\psi_n\}$ converges uniformly on any compact subsets of Ω to a holomorphic map $g : \Omega \rightarrow M_{H_\infty}$. In the other hand, since Ω is taut we can assume that $\{\psi_n^{-1}\}$ converges also uniformly on any compact subset of M_{H_∞} to a holomorphic map $\tilde{g} : M_{H_\infty} \rightarrow M_H$. Therefore it follows from Proposition 1 that g is biholomorphic, and hence Ω is biholomorphically equivalent to M_{H_∞} . \square

Remark 2. $\text{dist}(\eta_n, \partial M_H) \approx \epsilon_n := |\rho(\eta_n)|$.

Remark 3. i) Let $\{\eta_n\}$ be a sequence in M_H which converges tangentially to order $2m$ to $(0, 0)$. Set $\epsilon_n := |\rho(\eta_n)| \approx |\eta_{n1}|^{2m}$. Then we have

$$\begin{aligned} |\text{Re } \eta_{n2}| &= |\epsilon_n + H(\eta_{n1})| \\ &\lesssim |\eta_{n1}|^{2m}. \end{aligned}$$

ii) Suppose that $\{\eta_n\}$ is a sequence in M_H which converges tangentially to order $< 2m$ to $(0, 0)$. Then we have $|\eta_{n1}|^{2m} = o(\epsilon_n)$ and we thus obtain the following estimate

$$\begin{aligned} |\text{Re } \eta_{n2}| &= |\epsilon_n + H(\eta_{n1})| \\ &\approx |\epsilon_n|. \end{aligned}$$

Lemma 2. If $\{\eta_n\} \subset M_H$ converges tangentially to order $2m$ to $(0, 0)$, then $\deg(H_\infty) = 2m$ and moreover M_{H_∞} is biholomorphically equivalent to M_H .

Proof. Since $\{\eta_n\}$ converges tangentially to order $2m$ to $(0, 0)$, it follows that $|\eta_{n1}|^{2m} \approx \epsilon_n \approx d(\eta_n, \partial\Omega)$. Let $a_{j,\bar{q}}(\eta_n) := \frac{H_{j,\bar{q}}(\eta_{n1})\tau(\eta_n)^{j+q}}{(j+q)!\epsilon_n}$ for each $j, q > 0$ with $j+q \leq 2m$. Then we have the following estimate

$$|a_{j,\bar{q}}(\eta_n)| \lesssim \frac{|\eta_{n1}|^{2m-j-q}\tau(\eta_n)^{j+q}}{(j+q)!\epsilon_n} \lesssim \left(\frac{\tau(\eta_n)}{|\eta_{n1}|}\right)^{j+q}.$$

Since $\|H_{\eta_n}\| = 1$, we have $\tau(\eta_n) \gtrsim |\eta_{n1}| \approx \epsilon_n^{1/(2m)}$, and therefore $\tau(\eta_n) \approx \epsilon_n^{1/(2m)}$. This implies that $\deg(H_\infty) = 2m$. Without loss of generality we can assume that $\lim \frac{\eta_{n1}}{\epsilon_n^{1/(2m)}} = \alpha$ and $\lim \frac{\tau(\eta_n)}{\epsilon_n^{1/(2m)}} = \beta$. We note that

$$\begin{aligned} a_{j,\bar{q}}(\eta_n) &= \frac{H_{j,\bar{q}}(\eta_{n1})\tau(\eta_n)^{j+q}}{(j+q)!\epsilon_n} \\ &= \frac{1}{(j+q)!} \left(\frac{\tau(\eta_n)}{\epsilon_n^{1/(2m)}}\right)^{j+q} H_{j,\bar{q}}\left(\frac{\eta_{n1}}{\epsilon_n^{1/(2m)}}\right) \end{aligned}$$

for any $j, q > 0$. Then we obtain $\lim a_{j,\bar{q}}(\eta_n) = \frac{1}{(j+q)!} \beta^{j+q} H_{j,\bar{q}}(\alpha) w_1^j \bar{w}_1^q$ for each $j, q > 0$; hence

$$\begin{aligned} H_\infty(w_1) &= \sum_{j,q>0} \frac{1}{(j+q)!} \beta^{j+q} H_{j,\bar{q}}(\alpha) w_1^j \bar{w}_1^q \\ &= H(\alpha + \beta w_1) - H(\alpha) - 2\operatorname{Re} \sum_{j=1}^{2m} \frac{H_j(\alpha)}{j!} (\beta w_1)^j. \end{aligned}$$

So, the holomorphic map given by

$$\begin{cases} t_2 = w_2 - H(\alpha) - 2 \sum_{j=1}^{2m} \frac{H_j(\alpha)}{j!} (\beta w_1)^j \\ t_1 = \alpha + \beta w_1 \end{cases}$$

is biholomorphic from M_{H_∞} onto M_H . \square

Lemma 3. *If $\{\eta_n\} \subset M_H$ converges tangentially to order $< 2m$ to $(0,0)$, then $H_\infty = H$.*

Proof. It is easy to see that $\tau(\eta_n) \lesssim \epsilon_n^{1/(2m)}$. On the other hand, since $|\eta_{n1}|^{2m} = o(|\epsilon_n|)$, we have for $j, q \in \mathbb{N}$ with $j, q > 0, j+q < 2m$ that

$$\begin{aligned} |a_{j,\bar{q}}(\eta_n)| &\lesssim \frac{|\eta_{n1}|^{2m-j-q} \epsilon_n^{(j+q)/(2m)}}{(j+q)! \epsilon_n} \\ &\lesssim \left(\frac{|\eta_{n1}|^{2m}}{\epsilon_n} \right)^{\frac{2m-j-q}{2m}}. \end{aligned}$$

Therefore $\lim a_{j,\bar{q}}(\eta_n) = 0$ for any $j, q > 0$ with $j+q < 2m$, and thus $H_\infty = H$. Hence, the proof is complete. \square

Proof of Theorem 1. Let Ω and $\{f_n\}$ be a domain and a sequence, respectively, as in Theorem 1. Then, after a change of coordinates, we can assume that $p = (0,0)$ and

$$\Omega \cap U = \{(z_1, z_2) \in U : \rho(z_1, z_2) = \operatorname{Re} z_2 + H(z_1) < 0\}.$$

Moreover, we may also assume that $\eta_n := f_n(q) \in U \cap M_H$ for all $n \in \mathbb{N}^*$. Therefore, it follows from Proposition 2, Lemma 2, and Lemma 3 that Ω is biholomorphically equivalent to M_H , which finishes the proof. \square

In the case that $\{\eta_n\}$ converges tangentially to order $> 2m$ to $(0,0)$, we obtain the following proposition.

Proposition 3. *Let $\{\eta_n\} \subset M_H$ be a sequence which converges tangentially to order $> 2m$ to $(0,0)$. If there exist $j, q > 0$ with $j+q < 2m$ such that*

$$\left| \frac{\partial^{j+q} H}{\partial z_1^j \partial \bar{z}_1^q}(\eta_{n1}) \right| \approx |\eta_{n1}|^{2m-j-q},$$

then $\tau(\eta_n) = o(\epsilon_n^{1/(2m)})$, and thus $\deg(H_\infty) < 2m$.

Proof. Suppose otherwise that $\tau(\eta_n) \approx \epsilon_n^{1/(2m)}$. Then since $\epsilon_n = o(|\eta_{n1}|^{2m})$, one gets

$$\begin{aligned} |a_{j,\bar{q}}(\eta_n)| &\approx \frac{|\eta_{n1}|^{2m-j-q} \epsilon_n^{(j+q)/(2m)}}{(j+q)! \epsilon_n} \\ &\approx \left(\frac{|\eta_{n1}|^{2m}}{\epsilon_n} \right)^{\frac{2m-j-q}{2m}}. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} |a_{j,\bar{q}}(\eta_n)| = +\infty,$$

which is a contradiction. Thus, the proof is complete. \square

Example 1. Let $E_{1,2} := \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re} z_2 + |z_1|^4 < 0\}$. Then the sequence $\{(1/\sqrt[4]{n}, -2/n)\}$ converges tangentially to order 4 to $(0, 0)$. But the sequence $\{(1/\sqrt[4]{n}, -1/n - 1/n^2)\}$ converges tangentially to order 8 to $(0, 0)$.

Let $\rho(z_1, z_2) = \operatorname{Re} z_2 + |z_1|^4$ and let $\eta_n = (1/\sqrt[4]{n}, -1/n - 1/n^2)$ for every $n \in \mathbb{N}^*$. We see that $\rho(\eta_n) = -1/n - 1/n^2 + 1/n = -1/n^2 \approx -\operatorname{dist}(\eta_n, \partial\Omega)$. Set $\epsilon_n = |\rho(\eta_n)| = 1/n^2$. Then

$$\begin{aligned} \rho(z_1, z_2) &= \operatorname{Re}(z_2) + \left| \frac{1}{\sqrt[4]{n}} + z_1 - \frac{1}{\sqrt[4]{n}} \right|^4 \\ &= \operatorname{Re}(z_2) + \frac{1}{n} + \left| z_1 - \frac{1}{\sqrt[4]{n}} \right|^4 + \frac{1}{\sqrt{n}} (2\operatorname{Re}(z_1 - \frac{1}{\sqrt[4]{n}}))^2 + \frac{4}{\sqrt[4]{n}} \left| z_1 - \frac{1}{\sqrt[4]{n}} \right|^2 \operatorname{Re}(z_1 - \frac{1}{\sqrt[4]{n}}) \\ &\quad + \frac{4}{\sqrt{n}} \frac{1}{\sqrt[4]{n}} \operatorname{Re}(z_1 - \frac{1}{\sqrt[4]{n}}) + \frac{2}{\sqrt{n}} \left| z_1 - \frac{1}{\sqrt[4]{n}} \right|^2 \\ &= \operatorname{Re}(z_2) + \frac{1}{n} + \frac{4}{\sqrt{n}\sqrt[4]{n}} \operatorname{Re}(z_1 - \frac{1}{\sqrt[4]{n}}) + \frac{2}{\sqrt{n}} \operatorname{Re}((z_1 - \frac{1}{\sqrt[4]{n}})^2) + \left| z_1 - \frac{1}{\sqrt[4]{n}} \right|^4 \\ &\quad + \frac{4}{\sqrt{n}} \left| z_1 - \frac{1}{\sqrt[4]{n}} \right|^2 + \frac{4}{\sqrt[4]{n}} \left| z_1 - \frac{1}{\sqrt[4]{n}} \right|^2 \operatorname{Re}(z_1 - \frac{1}{\sqrt[4]{n}}). \end{aligned}$$

A direct calculation shows that $\tau_n := \tau(\eta_n) = \frac{1}{2n^{3/4}}$ for all $n = 1, 2, \dots$ and thus the automorphism ϕ_{η_n} is given by

$$\begin{aligned} \phi_{\eta_n}^{-1}(w_1, w_2) &= \left(\frac{1}{\sqrt[4]{n}} + \tau_n w_1, \epsilon_n w_2 - \frac{1}{n} - \frac{4}{\sqrt{n}\sqrt[4]{n}} \tau_n \operatorname{Re}(w_1) - \frac{2}{\sqrt{n}} \tau_n^2 \operatorname{Re}(w_1^2) \right); \\ \epsilon_n^{-1} \rho \circ \phi_{\eta_n}^{-1}(w_1, w_2) &= \epsilon_n^{-1} \rho \left(\frac{1}{\sqrt[4]{n}} + \tau_n w_1, \epsilon_n w_2 - \frac{1}{n} - \frac{4}{\sqrt{n}\sqrt[4]{n}} \tau_n \operatorname{Re}(w_1) - \frac{2}{\sqrt{n}} \tau_n^2 \operatorname{Re}(w_1^2) \right) \\ &= \operatorname{Re}(w_2) + \frac{1}{16n} |w_1|^4 + |w_1|^2 + \frac{1}{2\sqrt[4]{n}} |w_1|^2 \operatorname{Re}(w_1). \end{aligned}$$

We now show that there do not exist a sequence $\{f_n\} \subset \operatorname{Aut}(E_{1,2})$ and $a \in E_{1,2}$ such that $\eta_n = f_n(a) \rightarrow (0, 0) \in \partial E_{1,2}$ as $n \rightarrow \infty$. Indeed, suppose that there exist such a sequence $\{f_n\}$ and such a point $a \in E_{1,2}$. Then by Proposition 2, $E_{1,2}$ is biholomorphically equivalent to the following domain $D := \{(w_1, w_2) \in \mathbb{C}^2 : \operatorname{Re} w_2 + |w_1|^2 < 0\} \simeq \mathbb{B}^2$. It is impossible.

4. AUTOMORPHISM GROUP OF A MODEL IN \mathbb{C}^2

In this section, we consider a model

$$M_H := \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re} z_2 + H(z_1) < 0\},$$

where

$$H(z_1) = \sum_{j=1}^{2m-1} a_{2m-j} z_1^j z_1^{2m-j} = a_m |z_1|^{2m} + 2 \sum_{j=1}^{m-1} |z_1|^{2j} \operatorname{Re}(a_j z_1^{2m-2j}) \quad (1)$$

is a nonzero real valued homogeneous polynomial of degree $2m$, with $a_j \in \mathbb{C}$ and $a_j = \overline{a_{2m-j}}$. We will give the explicit description of $\operatorname{Aut}(M_H)$.

The D'Angelo type of ∂M_H is given by the following.

Lemma 4. $\tau(\partial M_H, (\alpha, -H(\alpha) + it)) = m_\alpha$ for all $\alpha \in \mathbb{C}$ and for all $t \in \mathbb{R}$, where

$$m_\alpha = \min\{j + q \mid j, q > 0, \frac{\partial^{j+q} H(\alpha)}{\partial z_1^j \partial \bar{z}_1^q} \neq 0\}.$$

Proof. By the following change of variables

$$\begin{cases} w_2 = z_2 + H(\alpha) + 2 \sum_{j=1}^{2m} \frac{H_j(\alpha)}{j!} (z_1 - \alpha)^j \\ w_1 = z_1 - \alpha, \end{cases}$$

the defining function for M_H is now given by

$$\rho(w_1, w_2) = \operatorname{Re} w_2 + \sum_{j,q>0} \frac{H_{j,\bar{q}}(\alpha)}{(j+q)!} w_1^j \bar{w}_1^q.$$

By a computation, we get $\tau(\partial M_H, (\alpha, -H(\alpha) + it)) = m_\alpha$, and thus the proof is complete. \square

Let $P_k(\partial M_H)$ the set of all points in ∂M_H of D'Angelo type k (k is either a positive integer or infinity). Let us denote by $\Gamma := \{(z_1, it) \mid t \in \mathbb{R}, z_1 \in \mathbb{C} \text{ with } \operatorname{Re}(e^{i\nu} z_1) = 0\}$ if $H(z_1) = a(\operatorname{Re}(e^{i\nu} z_1))^{2m}$ for some $a \in \mathbb{R}^*$ and for some $\nu \in [0, 2\pi)$ and by $\Gamma := \{(0, it) \mid t \in \mathbb{R}\}$ if otherwise.

Lemma 5. If $m \geq 2$, then $P_{2m}(\partial M_H) = \Gamma$ and $\tau(\partial M_H, p) < 2m$ for all $p \in \partial M_H \setminus \Gamma$.

Proof. It is not hard to show that $\Gamma \subset P_{2m}(\partial M_H)$. Now let $p = (\alpha, -H(\alpha) + it)$ ($\alpha \neq 0$) be any boundary point in $\partial M_H \setminus \Gamma$. By Lemma 4, we see that $\tau(\partial M_H, p) = m_\alpha \leq 2m$. We will prove that $\tau(\partial M_H, p) < 2m$. Indeed, suppose that, on the contrary, $\tau(\partial M_H, p) = m_\alpha = 2m$. This implies that $H_{j,\bar{q}}(\alpha) = 0$ for all $j, q > 0$ and $j + q < 2m$ and thus $H_{1,\bar{1}}(\alpha + z_1) = H_{1,\bar{1}}(z_1)$ for all $z_1 \in \mathbb{C}$. Let $g(x, y) := H_{1,\bar{1}}(x + iy)$ for all $z_1 = x + iy \in \mathbb{C}$. By a change of affine coordinates in \mathbb{C} , we may assume that $\alpha = (1, 0)$ and thus $f(x + 1, y) = f(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Hence, for each $y \in \mathbb{R}$ $f(x, y)$ is a periodic polynomial in x , and thus $f(x, y)$ does not depend on x , i.e., $f(x, y) = \beta y^{2m-2}$ for some $\beta \in \mathbb{R}$.

Therefore by the above, we conclude that $H_{1,\bar{1}}(z_1) = a(\operatorname{Re}(e^{i\nu} z_1))^{2m-2}$ for some $a \in \mathbb{R}^*$ and for some $\nu \in [0, 2\pi)$ and α satisfies $\operatorname{Re}(e^{i\nu} \alpha) = 0$. It is easy to show that $H(z_1) = c(\operatorname{Re}(e^{i\nu} z_1))^{2m-2}$ for some $c \in \mathbb{R}^*$ and $(\alpha, it) \in \Gamma$, which is impossible. Thus the proof is complete. \square

We recall the following lemma, proved by F. Berteloot (see [7]), which is the main ingredient in the proof of Theorem 2.

Lemma 6 (F. Berteloot). *Let $Q \in \mathcal{P}_{2m}$ and $H \in \mathcal{H}_{2m}$. Suppose that $\psi : M_H \rightarrow M_Q$ is a biholomorphism. Then there exist $t_0 \in \mathbb{R}$ and $z_0 \in \partial M_Q$ such that ψ and ψ^{-1} extend to be holomorphic in neighborhoods of $(0, it_0)$ and z_0 , respectively. Moreover, the homogeneous part of higher degree in Q is equal to $\lambda H(e^{i\nu} z)$ for some $\lambda > 0$ and $\nu \in [0, 2\pi)$.*

Proof. According to [2], there exists a holomorphic function ϕ on M_Q which is continuous on $\overline{M_Q}$ such that $|\phi| < 1$ for $z \in M_Q$ and tends to 1 at infinity. Let $\psi : M_H \rightarrow M_Q$ be a biholomorphism. We claim that there exists $t_0 \in \mathbb{R}$ such that $\lim_{x \rightarrow 0^-} \inf |\psi(0', x + it_0)| < +\infty$. Indeed, if this would not be the case, the function $\phi \circ \psi$ would be equal to 1 on the half plane $\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re} z_2 < 0, z_1 = 0\}$ and this is impossible since $|\phi| < 1$ for $|z| \gg 1$. Therefore, we may assume that there exists a sequence $x_k < 0$ such that $\lim x_k = 0$ and $\lim \psi(0, x_k + it_0) = z_0 \in \partial M_Q$. It is proved in [6] that under these circumstances ψ extends homeomorphically to ∂M_H on some neighbourhood of $(0, it_0)$. Then the result of Bell (see [4]) shows that this extension is actually diffeomorphic. Moreover, it follows from [5, Theorem 3] (see also [12, 24]) ψ and ψ^{-1} extend to be holomorphic in neighborhoods of $(0, it_0)$ and z_0 , respectively. Therefore, the conclusion follows easily. \square

Now we recall two basic integer valued invariants used in the normal form construction in [21]. Let $l = m_0 < m_1 < \dots < m_p \leq m$ be indices in (1) for which $a_{m_i} \neq 0$. Denote by L the greatest common divisor of $2m - 2m_0, 2m - 2m_1, \dots, 2m - 2m_p$. If $l = m$, then $H(z_1) = a_m |z_1|^{2m}$ ($a_m > 0$) and it is known that M_H is biholomorphically equivalent to the domain

$$E_{1,m} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2|^2 + |z_1|^{2m} < 1\}.$$

The automorphism group of $E_{1,m}$ is well-known (see [15, Example 9, p. 20]). So, in what follows we only consider the case $l < m$. Moreover, we consider the model $\Omega_m = \{(z_1, z_2) \in \mathbb{C} : \operatorname{Re} z_2 + (\operatorname{Re} z_1)^{2m} < 0\}$ and others which are not biholomorphically equivalent to it. We notice that the constant $L = 2$ for the model Ω_m

Lemma 7. *$H(\exp(i\theta)z_1) = H(z_1)$ for all $z_1 \in \mathbb{C}$ if and only if $\exp(i\theta)$ is an L -root of unity.*

Proof. We have

$$H(\exp(i\theta)z_1) = a_m |z_1|^{2m} + 2 \sum_{j=0}^p \left(|z_1|^{2j} \operatorname{Re} \left\{ a_{m_j} \exp(i(2m - 2m_j)\theta) z_1^{2m-2m_j} \right\} \right)$$

for all $z_1 \in \mathbb{C}$. Hence, we conclude that $H(\exp(i\theta)z_1) = H(z_1)$ for all $z_1 \in \mathbb{C}$ if and only if $\exp(i(2m - 2m_j)\theta) = 1$ for every $j = 0, \dots, p$, which proves the assertion. \square

Proof of Theorem 2. For $t \in \mathbb{R}$, $\lambda > 0$, and any L -root of unity $\exp(i\theta)$, consider the mappings

$$\begin{aligned} T_t^1 : (z_1, z_2) &\mapsto (z_1 + it, z_2); \\ T_t^2 : (z_1, z_2) &\mapsto (z_1, z_2 + it); \\ R_\theta : (z_1, z_2) &\mapsto (e^{i\theta} z_1, z_2); \\ S_\lambda : (z_1, z_2) &\mapsto (\lambda z_1, \lambda^{2m} z_2). \end{aligned}$$

It is easy to check that $T_t^2, R_\theta, S_\lambda$ are in $\text{Aut}(M_H)$ and moreover $T_t^1 \in \text{Aut}(M_H)$ if $H(z_1) = (\text{Re } z_1)^{2m}$ for all $z_1 \in \mathbb{C}$. Now let $f = (f_1, f_2)$ be any biholomorphism of M_H . It follows from Lemma 6 that there exist boundary points $p \in \Gamma$ and $q \in \Gamma$ such that f and f^{-1} extend to be holomorphic in neighborhoods of p and q , respectively, and $f(p) = q$. Replacing f by its composition with reasonable translations T_t^2, T_t^1 , we may assume that $p = q = (0, 0)$, and there exist neighborhoods U_1 and U_2 of $(0, 0)$ such that $U_2 \cap \partial M_H = f(U_1 \cap \partial M_H)$, and f and f^{-1} are holomorphic in U_1 and U_2 , respectively. Moreover, f is a local CR diffeomorphism of between $U_1 \cap \partial M_H$ and $U_2 \cap \partial M_H$.

Let us denote by $\mathcal{H} = \{z \in \mathbb{C} : \text{Re } z < 0\}$. We now define $g_1(z_2) := f_1(0, z_2)$ and $g_2(z_2) := f_2(0, z_2)$ for all $z_2 \in \mathcal{H}$. It follows from Lemma 4 that $f(U_1 \cap \Gamma) = U_2 \cap \Gamma$. Consequently, $g_1(it) = 0$ for all $-\epsilon_0 < t < \epsilon_0$ with $\epsilon_0 > 0$ small enough. By the Schwarz Reflection Principle and the Identity Theorem, we have $g_1(z_2) = 0$ for all $z_2 \in \mathcal{H}$. This also implies that $\text{Re } f_2(0, z_2) < 0$, and thus $g_2 \in \text{Aut}(\mathcal{H})$. Since $g_2(0) = 0$, it is known that $g_2(z_2) = \frac{\alpha z_2}{1 + i\beta z_2}$ for some $\alpha \in \mathbb{R}^*$ and $\beta \in \mathbb{R}$.

Now we are going to prove that f is biholomorphic between neighborhoods of the origin. To do this, it suffices to show that $J_f(0, 0) \neq 0$ (a similar proof shows that $J_{f^{-1}}(0, 0) \neq 0$). To derive a contradiction, we suppose that $J_f(0, 0) = 0$. By the above we can write

$$f(z_1, z_2) = (z_1 a(z_1, z_2), g_2(z_2) + z_1 b(z_1, z_2)),$$

where a and b are holomorphic functions defined on neighborhoods of $(0, 0)$, respectively. By shrinking U_1 if necessary, we can assume that a, b are defined on U_1 .

Take derivative of f at points $(0, z_2)$ we have

$$df(z_1, z_2) = \begin{pmatrix} a(z_1, z_2) & z_1 a_{z_1}(z_1, z_2) \\ b(z_1, z_2) & g_2'(z_2) + z_1 b_{z_2}(z_1, z_2) \end{pmatrix}.$$

Therefore we obtain $J_f(0, z_2) = a(0, z_2)g_2'(z_2)$ for every z_2 small enough. We note that $J_f(0, z_2) \neq 0$ for all $z_2 \in \mathcal{H}$, $g_2'(0) = \alpha \neq 0$, and $J_f(0, 0) = 0$. This implies that $a(z_1, z_2) = O(|z|)$.

Since $f(z_1, z_2) \in \overline{M_H} \cap U_2$ for all $(z_1, z_2) \in \overline{M_H} \cap U_1$,

$$\text{Re}(g_2(z_2) + z_1 b(z_1, z_2)) + H(z_1 a(z_1, z_2)) \leq 0$$

for all $(z_1, z_2) \in \overline{M_H} \cap U_1$. Because of the invariance of $\overline{M_H}$ under any map S_t ($t > 0$), one gets

$$\text{Re}(g_2(t^{2m} z_2) + t z_1 b(t z_1, t^{2m} z_2)) + H(t z_1 a(t z_1, t^{2m} z_2)) \leq 0 \quad (2)$$

for every $(z_1, z_2) \in \overline{M_H} \cap U_1$ and for every $t \in (0, 1)$.

Expand the function b into the Taylor series at the origin so that

$$b(z_1, z_2) = \sum_{j,k=0}^{\infty} b_{j,k} z_1^j z_2^k,$$

where $b_{j,k} \in \mathbb{C}$ for all $j, k \in \mathbb{N}$. Hence the equation (2) can be re-written as

$$\begin{aligned} \rho \circ f(t z_1, t^{2m} z_2) &= \text{Re} \left(\alpha \frac{t^{2m} z_2}{1 + i\beta t^{2m} z_2} + t z_1 \sum_{j,k=0}^{\infty} b_{j,k} (t z_1)^j (t^{2m} z_2)^k \right) \\ &\quad + H(t z_1 a(t z_1, t^{2m} z_2)) \leq 0 \end{aligned} \quad (3)$$

for every $(z_1, z_2) \in \overline{M_H} \cap U_1$ and for every $t \in (0, 1)$.

Now let us denote by $j_0 = \min\{j \mid b_{j,0} \neq 0\}$ if $b(z_1, 0) \not\equiv 0$ and $j_0 = +\infty$ if otherwise. We divide the argument into three cases as follows.

Case 1. $0 \leq j_0 \leq 2m - 2$. Note that we can choose $\delta_0 > 0$ and $\epsilon_0 > 0$ such that $H(z_1) < \epsilon_0$ for all $|z_1| < \delta_0$. Since $(-\epsilon_0, z_1) \in U \cap M_H$ for all $|z_1| < \delta_0$, taking $\lim_{t \rightarrow 0^+} \frac{1}{t^{j_0+1}} \rho \circ f(tz_1, t^{2m}\epsilon_0)$ we obtain $\operatorname{Re}(b_{j_0,0}z_1^{j_0+1}) \leq 0$ for all $|z_1| < \delta_0$, which leads to a contradiction.

Case 2. $j_0 = 2m - 1$. It follows from (3) that

$$\lim_{t \rightarrow 0^+} \frac{1}{t^{2m}} \rho \circ f(tz_1, t^{2m}z_2) = \operatorname{Re}(\alpha z_2 + b_{2m-1,0}z_1^{2m}) = 0$$

for all $(z_1, z_2) \in U_1$ with $\operatorname{Re} z_2 + H(z_1) = 0$. This implies that $H(z_1) = \operatorname{Re}(\frac{b_{2m-1,0}}{\alpha}z_1^{2m})$ for all $|z_1| < \delta_0$ with $\delta_0 > 0$ small enough. It is absurd since H contains no harmonic terms.

Case 3. $j_0 > 2m - 1$. Fix a point $(z_1, z_2) \in U \cap \partial M_H$ with $\operatorname{Re}(z_2) \neq 0$. From (3) one has

$$\lim_{t \rightarrow 0^+} \frac{1}{t^{2m}} \rho \circ f(tz_1, t^{2m}z_2) = \operatorname{Re}(\alpha z_2) = 0,$$

which is impossible.

Altogether, we conclude that f is a local biholomorphism between neighborhoods U_1 and U_2 of the origin satisfying $f(U_1 \cap \partial M_H) = U_2 \cap \partial M_H$. Therefore by [21, Corollary 5.3, p. 909] and the Identity Theorem, we have

$$f(z_1, z_2) = (\lambda e^{i\theta} z_1, \lambda^{2m} z_2)$$

for all $(z_1, z_2) \in M_H$, where $e^{i\theta}$ is an L -root of unity. Thus $f = S_\lambda \circ R_\theta$, and hence the proof is complete. \square

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